Coupling constant metamorphosis as an integrability-preserving transformation for general finite-dimensional dynamical systems and ODEs

Artur Sergyeyev

Mathematical Institute, Silesian University in Opava,
Na Rybníčku 1, 746 01 Opava, Czech Republic
E-mail: Artur.Sergyeyev@math.slu.cz

In the present paper we extend the multiparameter coupling constant metamorphosis, also known as the generalized Stäckel transform, from Hamiltonian dynamical systems to general finite-dimensional dynamical systems and ODEs. This transform interchanges the values of integrals of motion with the parameters these integrals depend on but leaves the phase space coordinates intact. Sufficient conditions under which the transformation in question preserves integrability and a simple formula relating the solutions of the original system to those of the transformed one are given.

Keywords: coupling constant metamorphosis; generalized Stäckel transform; dynamical systems; ODEs; integrals of motion; symmetries; Lax pairs; PDEs

1 Introduction

The modern theory of integrable finite-dimensional dynamical systems concentrates mostly on Hamiltonian systems. For the latter, the presence of symplectic or Poisson structure combined with the Liouville theorem leads to a number of important simplifications, in particular regarding the relationship among symmetries and integrals of motion. However, this also leads to certain restrictions because while considering various transformations of Hamiltonian systems one naturally wants the transformed systems to be Hamiltonian too. Such transformations are of interest for two reasons: they enable one to reduce the lists of integrable systems of certain form resulting from various classification procedures, and, sometimes, to relate new integrable systems to the known ones.

In the present paper we explore a somewhat surprising situation when the *absence* of need to preserve the Poisson or symplectic structure considerably *simplifies* things for general (non-Hamiltonian) dynamical systems including ODEs. This is precisely the case for the multiparameter coupling constant metamorphosis, also known as the generalized Stäckel transform, see [8, 5, 17, 3] and references therein. The said transform maps a set of integrals of motion in involution into another such set. In essence, it interchanges the *parameters* present in the integrals of motion with the *values* of those integrals, thus producing a new set of integrals of motion. Note that the dynamical variables, i.e., the phase space coordinates, are *not* affected by this transformation.

The above recipe works provided the original set of integrals of motion depends on some parameters in a nontrivial fashion, and enables one to produce a number of interesting new examples or, conversely, to relate certain new integrable systems to the known ones, cf. [17, 3]. However, in the Hamiltonian case there is a catch: the associated transformation for the equations of motion can only be written down upon restricting these equations to a common level surface of integrals of motion generating the transform under study and turns out to be a rather nontrivial reciprocal transformation, see [17] for details. We need this reciprocal transformation precisely because we want the transformed equations of motion to originate from the transformed Hamiltonian through the Poisson structure which should be left intact.

Below we extend the multiparameter generalized Stäckel transform to general continuous dynamical systems, for which there is no Poisson structure to preserve. In this case it is possible to jettison the reciprocal transformation described above and define the associated transformation of equations of motion in a much simpler fashion. These results are summarized in Theorem 1 and Corollary 2 for general continuous dynamical systems and the overdetermined systems of first order PDEs, respectively, and in Corollary 3 for ODEs. What is more, Proposition 1 shows that the transformed system inherits existence of a Lax representation from the original one. The most important advantage of abandoning the reciprocal transformations is a very simple relationship among solutions of the original system and those of the transformed one, see Remark 1 below.

As an aside, note that the requirement of presence of parameters in the dynamical systems under study and in their integrals of motion is not as restrictive as it may seem at first glance because the parameters can often be introduced by hand, e.g. through changes of dependent and independent variables, and often simple transformations like translation or rescaling of dependent variables enable one to produce interesting examples, see e.g. Examples 2, 3 and 5 below.

2 Coupling constant metamorphosis for general dynamical systems

Consider an open domain $M \subset \mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and a dynamical system on M,

$$dx^\alpha/dt = X^\alpha(t, x^1, \dots, x^n, a_1, \dots, a_k), \quad \alpha = 1, \dots, n, \tag{1}$$

where x^α are coordinates on M, and $a_j \in \mathbb{K}$ are parameters. Thus, *de facto* we have a k-parametric family of dynamical systems but for the ease of writing we shall refer to it below as if it were a single dynamical system. The same convention will apply to its symmetries, integrals of motion, Lax matrices, etc. We deliberately use the local setting instead of the more global one (vector fields on manifolds, etc.) because of the subsequent necessity to invoke the implicit function theorem which almost inevitably forces one to consider things locally, as explained below. In what follows all objects are tacitly assumed to be smooth in all of their arguments.

To (1) we can naturally associate a vector field on M which depends on the parameters $t, a_1, \dots, a_k$,

$$X = \sum_{\alpha=1}^{n} X^\alpha \frac{\partial}{\partial x^\alpha}.$$

Recall that a (smooth) function $f = f(t, x^1, \dots, x^n, a_1, \dots, a_k)$ is an *integral of motion* (or a first integral) for the dynamical system (1) if we have

$$\partial f/\partial t + X(f) = 0. \tag{2}$$

Consider another dynamical system on M,

$$dx^\alpha/d\tau = Y^\alpha(t, x^1, \dots, x^n, a_1, \dots, a_k), \quad \alpha = 1, \dots, n, \tag{3}$$

and the associated vector field on M,

$$Y = \sum_{\alpha=1}^{n} Y^\alpha \frac{\partial}{\partial x^\alpha}.$$

Recall that Y is a *symmetry* for (1) if we have

$$\frac{d^2 x^\alpha}{dt d\tau} = \frac{d^2 x^\alpha}{d\tau dt}, \quad \alpha = 1, \dots, n,$$

where the derivatives are computed by virtue of (1) and (3), or equivalently,

$$\partial Y/\partial t + [X, Y] = 0. \tag{4}$$

Here and below $[\cdot, \cdot]$ stands for the Lie bracket of vector fields (i.e., the usual commutator of differential operators) unless otherwise explicitly stated.

Let (1) have k functionally independent integrals $I_1, \dots, I_k$ such that

$$\det\left(\|\partial I_i/\partial a_j\|_{i,j=1,\dots,k}\right) \neq 0. \tag{5}$$

Then by the implicit function theorem the equations

$$I_i(x, t, a_1, \dots, a_k) = b_i, \quad i = 1, \dots, k, \quad x \in M, \tag{6}$$

where b_i are constants, are (in general only locally, see the discussion in the beginning of the next section) uniquely solvable w.r.t. a_i. Denote the solution in question as follows:

$$a_i = \tilde{I}_i(x, t, b_1, \dots, b_k), \quad i = 1, \dots, k, \quad x \in M. \tag{7}$$

The reason for this notation will become clear in a moment.

If K is a geometrical object on M (a function, a vector field, a tensor field, a differential form, etc.) which may depend on the parameters a_i, then $\tilde{K}$ will stand for the geometrical object obtained from K by substituting $\tilde{I}_i$ for a_i for all $i = 1, \dots, k$. We shall write this as

$$\tilde{K} = K|_{a_1 = \tilde{I}_1, \dots, a_k = \tilde{I}_k}.$$

Thus, for instance, $\tilde{X}$ is a vector field obtained from X by substituting $\tilde{I}_i$ for a_i for all $i = 1, \dots, k$:

$$\tilde{X} = \sum_{\alpha=1}^{n} X^{\alpha}|_{a_1=\tilde{I}_1,\dots,a_k=\tilde{I}_k} \frac{\partial}{\partial x^{\alpha}},$$

and the associated dynamical system reads

$$dx^{\alpha}/dt = \tilde{X}^{\alpha}(t, x^1, \dots, x^n, b_1, \dots, b_k), \quad \alpha = 1, \dots, n. \tag{8}$$

The only exception from the above notational convention is made for the functions $\tilde{I}_i$ which are not related to I_i in the fashion described above. The reason for this apparent discrepancy is that $\tilde{I}_i$ turn out to be first integrals for $\tilde{X}$, see Theorem 1 below.

Note that $\tilde{X}$ and $\tilde{I}_i$ depend on the parameters $b_1, \dots, b_k$, and that $\tilde{I}_i$ obviously are functionally independent.

In analogy with [17], we shall refer to the procedure of passing from (1), X and I_i to (8), $\tilde{X}$ and $\tilde{I}_i$ as to the *k-parameter generalized Stäckel transform* (or as to the *k-parameter coupling constant metamorphosis*) generated by $I_1, \dots, I_k$. We shall also say that (1) is *Stäckel-equivalent* to (8).

Just as in the Hamiltonian setting [17], we have the following duality: when applied to (8), the k-parameter generalized Stäckel transform generated by $\tilde{I}_1, \dots, \tilde{I}_k$ brings us back to the integrals $I_1, \dots, I_k$ and the system (1) we have started with.

As a final remark, note that the above construction admits the following geometric interpretation.

Consider the extended phase space $N = M \times P$, where $P \subset \mathbb{K}^k$, an open subset of $\mathbb{K}^k$, is the space where our parameters live: $P \ni \vec{a} = (a_1, \dots, a_k)^T$; here and below the superscript T indicates the transposed matrix.

The dynamical system (1) can be naturally extended to N upon setting

$$da_i/dt = 0, \quad i = 1, \dots, k. \tag{9}$$

In other words, the extended dynamical system (1)+(9) on N is determined by *the same* vector field X as the original system (1) but X is now treated as a vector field on N. Conversely, the original system (1) is recovered from the extended one upon *fixing* the values of parameters a_i, $i = 1, \dots, k$.

The extended system under study admits, in addition to k integrals of motion I_j, $j = 1, \dots, k$, which are there by assumption, the 'obvious' integrals of motion a_j, $j = 1, \dots, k$ (for a moment, we ignore the possibility of existence of further integrals of motion for (1)). Thus, N foliates, at least locally, into common level surfaces of these $2k$ integrals. We shall denote this foliation by $\mathcal{F}_{2k}$.

Roughly speaking, the k-parameter generalized Stäckel transform, defined above, interchanges a_j and b_j as well as I_j and $\tilde{I}_j$, i.e., we choose a different way to parameterize the leaves of $\mathcal{F}_{2k}$.

The transform under study turns the *extended* system (1)+(9) into (8)+(10), where (10) reads

$$db_i/dt = 0, \quad i = 1, \dots, k. \tag{10}$$

While (1)+(9) and (8)+(10) coincide, at least locally (cf. the assumptions regarding the applicability of the implicit function theorem in the next section), on any given leaf of $\mathcal{F}_{2k}$, they are *different* when considered on N as a whole.

Our primary interest is, however, in the dynamical systems on the *original* phase space M rather than on the whole N, and this is where things become even more nontrivial: the transformed dynamical system (8) on M arises upon fixing the *new* parameters b_j rather than the old ones a_j, and (8) is a restriction onto M of the *transformed* extended dynamical system (8)+(10) on N.

3 Main results

The above considerations suggest that the generalized Stäckel transform should preserve a number of integrability attributes of the original system (e.g. symmetries, integrals of motion, etc.), and in the present section we state and prove the relevant results.

Here and below we use the following *blanket assumption*. We suppose that the domain M and the ranges of values of time t and of the parameters a_j and b_j, $j = 1, \dots, k$, are chosen so that the implicit function theorem ensures that the system (6) has a unique solution with respect to a_i, $i = 1, \dots, k$. In general this means making M and the ranges

in question sufficiently small because of the local nature of the implicit function theorem, but we deliberately do not fully spell out here the relevant conditions as there exists a number of situations when they would be too restrictive, e.g. when I_j are linear in all a_j, $j=1,\dots,k$ (or, more broadly, when we have explicit formulas for $\tilde{I}_j$ and one can employ these to specify M and the relevant ranges, as it is the case for the majority of interesting examples).

Theorem 1 *Let the dynamical system (1) have k functionally independent integrals I_i such that (5) holds. Define $\tilde{I}_i$ and the transformed quantities like $\tilde{X}$ as above.*

Then the following assertions hold:

i) the functions $\tilde{I}_i$, $i=1,\dots,k$, are functionally independent integrals for (8), and we have

$$\det\left(\|\partial\tilde{I}_i/\partial b_j\|_{i,j=1,\dots,k}\right)\neq 0; \tag{11}$$

ii) if $J_1,\dots,J_m$ is another set of integrals for (1) such that all integrals $I_1,\dots,I_k$, $J_1,\dots,J_m$ are functionally independent, then $\tilde{I}_1,\dots,\tilde{I}_k,\tilde{J}_1,\dots,\tilde{J}_m$ are functionally independent integrals for (8);

iii) if $Y_1,\dots,Y_r$ are linearly independent symmetries for (1) such that

$$Y_p(I_j)=0\quad \text{for all}\quad p=1,\dots,r\quad \text{and}\quad j=1,\dots,k, \tag{12}$$

then $\tilde{Y}_1,\dots,\tilde{Y}_r$ are linearly independent symmetries for (8), and

$$\tilde{Y}_p(\tilde{I}_j)=0\quad \text{for all}\quad p=1,\dots,r\quad \text{and}\quad j=1,\dots,k; \tag{13}$$

iv) if under the assumptions of (ii) and (iii) the symmetries $Y_1,\dots,Y_s$, where $s\leq r$, span an involutive distribution, i.e., $[Y_p,Y_q]=\sum\limits_{g=1}^{s}c_{pq}^g(a_1,\dots,a_k,I_1,\dots,I_k,J_1,\dots,J_m)Y_g$ for all $p,q=1,\dots,s$, then the symmetries $\tilde{Y}_1,\dots,\tilde{Y}_s$ also span an involutive distribution, $[\tilde{Y}_p,\tilde{Y}_q]=\sum\limits_{g=1}^{s}\tilde{c}_{pq}^g\tilde{Y}_g$, for all $p,q=1,\dots,s$, where $\tilde{c}_{pq}^g=c_{pq}^g(\tilde{I}_1,\dots,\tilde{I}_k,$ $b_1,\dots,b_k,\tilde{J}_1,\dots,\tilde{J}_m)$. If $c_{pq}^g\in\mathbb{K}$ are constants (in particular, they do not depend on a_j, $j=1,\dots,k$), and thus Y_g, $g=1,\dots,s$, form a Lie algebra, then $\tilde{Y}_g$, $g=1,\dots,s$, form an isomorphic Lie algebra.

Proof. Consider the identities

$$b_i\equiv I_i(x,\tilde{I}_1,\dots,\tilde{I}_k),\quad i=1,\dots,k,\quad x\in M, \tag{14}$$

that follow from (7). Obviously, $\partial b_i/\partial t+\tilde{X}(b_i)\equiv 0$, as b_i are constants, and hence acting by $\partial/\partial t+\tilde{X}$ on the left-hand side and the right-hand side of (14) yields

$$\begin{aligned}
0 \;\equiv\;& \left(\frac{\partial I_i}{\partial t}+\tilde{X}(I_i)\right)\Big|_{a_1=\tilde{I}_1,\dots,a_k=\tilde{I}_k}+\sum_{j=1}^{k}\left(\frac{\partial\tilde{I}_j}{\partial t}+\tilde{X}(\tilde{I}_j)\right)\left(\frac{\partial I_i}{\partial a_j}\right)\Big|_{a_1=\tilde{I}_1,\dots,a_k=\tilde{I}_k}\\
=\;& \left(\frac{\partial I_i}{\partial t}+X(I_i)\right)\Big|_{a_1=\tilde{I}_1,\dots,a_k=\tilde{I}_k}+\sum_{j=1}^{k}\left(\frac{\partial\tilde{I}_j}{\partial t}+\tilde{X}(\tilde{I}_j)\right)\left(\frac{\partial I_i}{\partial a_j}\right)\Big|_{a_1=\tilde{I}_1,\dots,a_k=\tilde{I}_k}\\
=\;& \sum_{j=1}^{k}\left(\frac{\partial\tilde{I}_j}{\partial t}+\tilde{X}(\tilde{I}_j)\right)\left(\frac{\partial I_i}{\partial a_j}\right)\Big|_{a_1=\tilde{I}_1,\dots,a_k=\tilde{I}_k},\quad i=1,\dots,k.
\end{aligned}$$

Here we used the chain rule and the fact that I_i are integrals for (1), and therefore by (2) we have

$$\frac{\partial I_i}{\partial t}+X(I_i)=0.$$

Using (5) we readily find that $\partial\tilde{I}_j/\partial t+\tilde{X}(\tilde{I}_j)=0$, $j=1,\dots,k$, so $\tilde{I}_j$ are indeed integrals for (8).

Eq.(11) follows from the implicit function theorem. The functional independence of $\tilde{I}_j$ is immediate. This completes the proof of part (i).

Next, I_j and J_s are integrals of motion for (1) by assumption, so $\partial I_j/\partial t + X(I_j) = 0$ and $\partial J_s/\partial t + X(J_s) = 0$, and using the chain rule shows that $\tilde{J}_s$ are integrals of motion for (8) as we have

$$\frac{\partial \tilde{J}_s}{\partial t} + \tilde{X}(\tilde{J}_s) = \left(\frac{\partial J_s}{\partial t} + X(J_s) + \sum_{j=1}^{k}\left(\frac{\partial I_j}{\partial t} + X(I_j)\right)\frac{\partial J_s}{\partial a_j}\right)\Bigg|_{a_1=\tilde{I}_1,\ldots,a_k=\tilde{I}_k} = 0, \quad s = 1, \ldots, m.$$

The functional independence of $\tilde{I}_1, \ldots, \tilde{I}_k$, $\tilde{J}_1, \ldots, \tilde{J}_m$ easily follows from that of $I_1, \ldots, I_k$, $J_1, \ldots, J_m$, and thus part (ii) is also proven.

In a similar fashion, further taking into account (12) and bearing in mind that $\partial Y_q/\partial t + [X, Y_q] = 0$ by assumption as Y_q are symmetries for (1), we obtain

$$\frac{\partial \tilde{Y}_q}{\partial t} + [\tilde{X}, \tilde{Y}_q] = \left(\frac{\partial Y_q}{\partial t} + [X, Y_q] + \sum_{j=1}^{k}\left(\left(\frac{\partial I_j}{\partial t} + X(I_j)\right)\frac{\partial Y_q}{\partial a_j} - Y_q(I_j)\frac{\partial X}{\partial a_j}\right)\right)\Bigg|_{a_1=\tilde{I}_1,\ldots,a_k=\tilde{I}_k} = 0, \quad q = 1, \ldots, r, \tag{15}$$

as desired. The linear independence of $\tilde{Y}_q$ readily follows from that of Y_q. Let us stress that if we drop the condition (12), the right-hand side of (15) is in general no longer obliged to vanish, and hence the quantities $\tilde{Y}_i$ will no longer be symmetries for $\tilde{X}$, cf. Example 1 below.

Finally, to prove (iv) we note that, in complete analogy with (15), we have

$$[\tilde{Y}_p, \tilde{Y}_q] = \left([Y_p, Y_q] + \sum_{j=1}^{k}\left(Y_p(I_j)\frac{\partial Y_q}{\partial a_j} - Y_q(I_j)\frac{\partial Y_p}{\partial a_j}\right)\right)\Bigg|_{a_1=\tilde{I}_1,\ldots,a_k=\tilde{I}_k}$$
$$= [Y_p, Y_q]|_{a_1=\tilde{I}_1,\ldots,a_k=\tilde{I}_k} = \sum_{s=1}^{r}\left(c_{pq}^s Y_s\right)|_{a_1=\tilde{I}_1,\ldots,a_k=\tilde{I}_k} = \sum_{s=1}^{r}\tilde{c}_{pq}^s \tilde{Y}_s, \quad p, q = 1, \ldots, r,$$

and the result follows. □

Informally, Theorem 1 states that if I_i and J_s are integrals and Y_j are symmetries for (1), and (5) and (12) hold, then $\tilde{I}_i$ and $\tilde{J}_s$ are integrals and $\tilde{Y}_j$ are symmetries for (8), the Lie algebra of symmetries $\tilde{Y}_j$ is 'essentially isomorphic' to that of Y_j, and (11) and (13) hold. By a slight abuse of terminology, it can be said that the multiparameter coupling constant metamorphosis (or the generalized Stäckel transform) preserves integrals of motion and symmetries that respect the generators I_j of the transform in question. Proceeding in the spirit of the proof of Theorem 1 it can be shown that the multiparameter generalized Stäckel transform preserves invariant curves and surfaces, Darboux multipliers, Jacobi multipliers and other similar structures. Thus, under certain technical assumptions the transformed system (8) inherits the integrability properties of (1).

It is now appropriate to recall the definition of extended integrability due to Bogoyavlenskij [4]:

Definition 1 ([4]) *A dynamical system (1) is* integrable in the broad sense *if it has m functionally independent integrals of motion $J_1, \ldots, J_m$, where $n > m \geq 0$, and $n-m$ linearly independent commuting symmetries $Y_1, \ldots, Y_{n-m}$ such that $Y_i(J_j) = 0$ for all $i = 1, \ldots, n-m$ and all $j = 1, \ldots, m$.*

Theorem 1 implies that the generalized Stäckel transform preserves extended integrability. Namely, the following assertion holds.

Corollary 1 *Let (1) be integrable in the broad sense, with the integrals J_j and symmetries Y_p as in Definition 1. Consider a k-parameter generalized Stäckel transform generated by the integrals I_j which are functions of J_s, $s = 1, \ldots, m$, i.e., $I_j = I_j(J_1, \ldots, J_m)$, $j = 1, \ldots, k$, and assume that (5) holds.*

Then the transformed system (8) is again integrable in the broad sense.

Proof. To prove this corollary it suffices to notice that we can construct from $J_1, \ldots, J_m$ a new set of functionally independent integrals, say, I_s, $s = 1, \ldots, m$, for (1), so that I_j for $s \leq k$ are precisely the generators of the Stäckel transform in question. Then by Theorem 1 the transformed quantities $\tilde{I}_s$, $s = 1, \ldots, m$, and $\tilde{Y}_j$, $j = 1, \ldots, n-m$, meet the requirements of Definition 1 for (8), if so do I_s and Y_j for (1), and the result follows. □

Remark 1. Unlike the case of Hamiltonian dynamical systems considered in [17], where the reciprocal transformation was involved, we have a very simple recipe for relating the solutions of (1) to those of (8). Namely, if $x^\alpha = \Xi^\alpha(t, a_1, \dots, a_k)$, $\alpha = 1, \dots, n$, is a solution for (1) then

$$x^\alpha = \Xi^\alpha(t, a_1, \dots, a_k)|_{t=\tilde{t}, a_1=\tilde{I}_1, \dots, a_k=\tilde{I}_k}, \alpha = 1, \dots, n, \tag{16}$$

is an *implicit* solution for (8).

In particular, if the formulas

$$x^\alpha = \Xi^\alpha(t, a_1, \dots, a_k, C_1, \dots, C_n), \quad \alpha = 1, \dots, n,$$

where $C_1, \dots, C_n$ are arbitrary constants, define a general solution for (1) then the formulas

$$x^\alpha = \Xi^\alpha(t, a_1, \dots, a_k, C_1, \dots, C_n)|_{t=\tilde{t}, a_1=\tilde{I}_1, \dots, a_k=\tilde{I}_k}, \quad \alpha = 1, \dots, n, \tag{17}$$

define an implicit general solution for (8). Using similar considerations one can also readily find out how an implicit or parametric (general or particular) solution of (1) transforms into an implicit or parametric (general or particular) solution of (8).

As a final remark, note that the multiparameter generalized Stäckel transform preserves (the existence of) the Lax representations. Namely, it is readily verified that the following assertion holds.

Proposition 1 *Let (1) admit a Lax representation of the form $dL/dt = [M, L]$, where L and M are $N \times N$ matrices that depend on $t, x^1, \dots, x^n, a_1, \dots, a_k$ and on a spectral parameter λ, and $[,]$ stands here for the commutator of matrices.*

Then the transformed dynamical system (8) possesses a Lax representation of the form $d\tilde{L}/dt = [\tilde{M}, \tilde{L}]$, where

$$\tilde{L} = L(t, x^1, \dots, x^n, a_1, \dots, a_k, \lambda)|_{a_1=\tilde{I}_1, \dots, a_k=\tilde{I}_k}, \quad \tilde{M} = M(t, x^1, \dots, x^n, a_1, \dots, a_k, \lambda)|_{a_1=\tilde{I}_1, \dots, a_k=\tilde{I}_k}.$$

4 Examples

To illustrate the above results, we start with the following easy example.

Example 1. Consider the one-component nonstationary dynamical system (i.e., a first-order ODE),

$$dx/dt = ax/((x-t)^2 + ax), \tag{18}$$

see equation 1.4.3-2.15 in [15].

For now let us work over $\mathbb{C}$. Then (18) admits an integral $I = \ln x + a/(t-x)$, and hence is integrable in the broad sense.

Setting $a_1 \equiv a$ and $b_1 \equiv b$ we readily find that $\tilde{I} = (b - \ln x)(t - x)$ is an integral of motion for the transformed equation

$$dx/dt = (b - \ln x)/(x - t + b - \ln x),$$

which therefore is also integrable in the broad sense.

As we have already noticed in Introduction, while many dynamical systems of interest do not involve parameters, we can often introduce the parameters 'by hand', e.g., through translation or rescaling of dependent variables.

Example 2. Consider the following dynamical system from Example 2.22 of [7]

$$dx/dt = -2x^2 + 2z, \quad dy/dt = -3xy, \quad dz/dt = 4xz - 2x(2x^2 - 9y^2).$$

which has an integral of motion of the form $z - x^2 + 3y^2$.

Upon rescaling the variable y, $y \to (a/3)^{1/2} y$, we obtain the system

$$dx/dt = -2x^2 + 2z, \quad dy/dt = -3xy, \quad dz/dt = 4xz - 2x(2x^2 - 3ay^2) \tag{19}$$

with an integral of motion

$$I = z - x^2 + ay^2.$$

Setting $a_1 \equiv a$ and $b_1 \equiv b$ and applying the general theory presented above we find that

$$\tilde{I} = \frac{b + x^2 - z}{y^2}$$

is an integral of motion for the transformed system

$$dx/dt = -2x^2 + 2z, \quad dy/dt = -3xy, \quad dz/dt = -2xz + 2x^3 + 6bx. \tag{20}$$

We now see that in the transformed system the right-hand side of the third equation is independent of y, so we have a decoupled subsystem for x and z,

$$dx/dt = -2x^2 + 2z, \quad dz/dt = -2xz + 2x^3 + 6bx,$$

i.e., (20) is, in a sense, indeed a somewhat simpler object than the original system (19). Moreover, (20) admits a symmetry $Y = y\partial/\partial y$. However, as Y does not preserve $\tilde{I}$, $Y(\tilde{I}) = -2\tilde{I} \neq 0$, this symmetry has no counterpart for the original system (19).

On the other hand, (20) is easily seen to have another integral $\tilde{I}_2 = x^2/2 + z + 2b\ln(y)$, and hence (20) is integrable by quadratures (and integrable in the broad sense).

Indeed, upon restriction onto the common level surface $\tilde{I} = C_1$ and $\tilde{I}_2 = C_2$ the system (20) boils down to a single ODE,

$$dy/dt = \mp y\sqrt{6\,(C_1 - b + C_2 y^2 - 2b\ln y)},$$

which is obviously integrable by quadratures, and hence so is (20).

Now, by Corollary 1 the above implies that (19) is also integrable in broad sense (and integrable by quadratures, as we can readily obtain the general solution for (19) from that of (20) using (16) and (17)). Note that the counterpart of $\tilde{I}_2$ for (19) reads

$$I_2 = x^2/2 + z + 2(ay^2 - x^2 + z)\ln(y)$$

Example 3. For a somewhat more elaborated example, consider system 9.25 from [9]

$$du/dt = -uv^2 + u + v, \quad dv/dt = u^2v - u - v, \quad dw/dt = v^2 - u^2$$

which has two integrals of motion, $u^2 + v^2 + \ln w^2$ and $w(uv - 1)$.

Upon rescaling $u \to (a_1)^{1/2}u$, $v \to (a_1)^{1/2}v$, $w \to w/a_2$, we obtain the system

$$du/dt = -a_1uv^2 + u + v, \quad dv/dt = a_1u^2v - u - v, \quad dw/dt = a_1a_2(v^2 - u^2)$$

with the integrals of motion

$$I_1 = a_1(u^2 + w^2) + \ln(w^2), \quad I_2 = a_2w(a_1uv - 1).$$

Consider the two-parametric generalized Stäckel transform generated by I_1 and I_2. We find that

$$\tilde{I}_1 = (b_1 - \ln(w^2))/(u^2 + w^2), \quad \tilde{I}_2 = -b_2(u^2 + w^2)/(uvw(\ln(w^2) - b_1) + (u^2 + w^2)w)$$

are integrals of motion for the system

$$\begin{aligned} du/dt &= -(b_1 - \ln(w^2))uv^2/(u^2 + w^2) + u + v, \\ dv/dt &= (b_1 - \ln(w^2))u^2v/(u^2 + w^2) - u - v, \\ dw/dt &= b_2(b_1 - \ln(w^2))(u^2 - v^2)/(uvw(\ln(w^2) - b_1) + (u^2 + w^2)w). \end{aligned}$$

5 Generalized Stäckel transform for overdetermined partial differential systems

Theorem 1 admits a natural generalization to the overdetermined systems of first-order PDEs which naturally arise e.g. in the study of zero-curvature representations, pseudopotentials and Bäcklund transformations for integrable (systems of) PDEs, cf. e.g. [1] and references therein.

Namely, consider an overdetermined system of first-order PDEs of the form

$$\frac{\partial x^\alpha}{\partial t^A} = X_A^\alpha(t^1,\dots,t^d,x^1,\dots,x^n,a_1,\dots,a_k), \quad \alpha=1,\dots,n, \quad A=1,\dots,d, \tag{21}$$

and assume that this system is in involution, i.e.,

$$\frac{\partial^2 x^\alpha}{\partial t^A \partial t^B} = \frac{\partial^2 x^\alpha}{\partial t^B \partial t^A}, \quad \alpha=1,\dots,n, \quad A,B=1,\dots,d, \tag{22}$$

where the derivatives are computed by virtue of (21), or equivalently,

$$\partial X_A/\partial t^B - \partial X_B/\partial t^A - [X_A,X_B]=0, \quad A,B=1,\dots,d. \tag{23}$$

Corollary 2 *Let (22) hold, and let (21) have k joint (i.e., such that $\partial I_j/\partial t^A + X_A(I_j) = 0$ for all A and j) functionally independent integrals $I_1,\dots,I_k$ such that (5) is satisfied.*

Then the following assertions hold:

i) the vector fields $\tilde X_A$, $A=1,\dots,d$, again commute: $[\tilde X_B,\tilde X_B]=0$, $a,b=1,\dots,d$, and hence the transformed system

$$\frac{\partial x^\alpha}{\partial t^A} = \tilde X_A^\alpha(t^1,\dots,t^d,x^1,\dots,x^n,b_1,\dots,b_k), \quad \alpha=1,\dots,n, \quad A=1,\dots,d,$$

is again in involution;

ii) the functions $\tilde I_i$, $i=1,\dots,k$, are functionally independent joint integrals for the vector fields $\tilde X_A$, $a=1,\dots,d$, and we have

$$\det\left(\|\partial \tilde I_i/\partial b_j\|_{i,j=1,\dots,k}\right)\neq 0;$$

iii) if $J_1,\dots,J_m$ is another set of joint integrals for X_A, $A=1,\dots,d$, such that all integrals $I_1,\dots,I_k,J_1,\dots,J_m$ are functionally independent, then $\tilde I_1,\dots,\tilde I_k,\tilde J_1,\dots,\tilde J_m$ are joint functionally independent integrals for $\tilde X_A$, $A=1,\dots,d$;

iv) if $Y_1,\dots,Y_r$ are linearly independent joint (i.e., $\partial Y_q/\partial t^A + [X_A,Y_q]=0$ for all A and q) symmetries for X_A, $A=1,\dots,d$, such that

$$Y_p(I_j)=0 \quad \text{for all} \quad p=1,\dots,r \quad \text{and} \quad j=1,\dots,k,$$

then $\tilde Y_1,\dots,\tilde Y_r$ are linearly independent joint symmetries for $\tilde X_A$, $A=1,\dots,d$, and

$$\tilde Y_p(\tilde I_j)=0 \quad \text{for all} \quad p=1,\dots,r \quad \text{and} \quad j=1,\dots,k;$$

v) if under the assumptions of (iii) and (iv) the symmetries $Y_1,\dots,Y_s$, where $s\le r$, span an involutive distribution, i.e., $[Y_p,Y_q]=\sum\limits_{g=1}^{s} c_{pq}^g(a_1,\dots,a_k,I_1,\dots,I_k,J_1,\dots,J_m)Y_g$ for all $p,q=1,\dots,s$, then the symmetries $\tilde Y_1,\dots,\tilde Y_s$ also span an involutive distribution, i.e., $[\tilde Y_p,\tilde Y_q]=\sum\limits_{g=1}^{s}\tilde c_{pq}^g\tilde Y_g$, for all $p,q=1,\dots,s$, where $\tilde c_{pq}^g = c_{pq}^g(\tilde I_1,\dots,\tilde I_k,$ $b_1,\dots,b_k,\tilde J_1,\dots,\tilde J_m)$. If $c_{pq}^g\in\mathbb{K}$ are constants (in particular, they do not depend on a_j, $j=1,\dots,k$), and thus Y_g, $g=1,\dots,s$, form a Lie algebra, then $\tilde Y_g$, $g=1,\dots,s$, form an isomorphic Lie algebra.

Stating the counterpart of Proposition 1 for (21) is left as an exercise for the reader.

6 Applications to ODEs

Consider an ODE resolved with respect to the highest-order derivative:

$$d^m u/dz^m = F\left(z, u, du/dz, \dots, d^{m-1}u/dz^{m-1}, a_1, \dots, a_k\right). \tag{24}$$

Let $n = m$, and put

$$x^1 = u, \quad x^2 = du/dz, \quad \dots, \quad x^m = d^{m-1}u/dz^{m-1}. \tag{25}$$

Consider a dynamical system

$$dx^1/dt = x^2, \quad dx^2/dt = x^3, \quad \dots, dx^{m-1}/dt = x^m, \quad dx^m/dt = f(t, x^1, \dots, x^m, a_1, \dots, a_k); \tag{26}$$

$$\text{here } f(t, x^1, \dots, x^m, a_1, \dots, a_k) = F\left(z, u, \frac{du}{dz}, \dots, \frac{d^{m-1}u}{dz^{m-1}}, a_1, \dots, a_k\right)\Big|_{z=t, u=x^1, du/dz=x^2, \dots, d^{m-1}u/dz^{m-1}=x^m}.$$

It is well known that the dynamical system (26) is equivalent to (24), and we can readily apply the result of Theorem 1 to (26).

What is more, it is immediate that upon applying the multiparameter generalized Stäckel transform to (26) we obtain the system of the same kind, that is,

$$\begin{aligned} &dx^1/dt = x^2, \quad dx^2/dt = x^3, \quad \dots, \\ &dx^{m-1}/dt = x^m, \quad dx^m/dt = \tilde f(t, x^1, \dots, x^m, b_1, \dots, b_k), \end{aligned} \tag{27}$$

which is, through (25), equivalent to an ODE of the form

$$\frac{d^m u}{dz^m} = \tilde F\left(z, u, \frac{du}{dz}, \dots, \frac{d^{m-1}u}{dz^{m-1}}, b_1, \dots, b_k\right), \tag{28}$$

where

$$\tilde F\left(z, u, \frac{du}{dz}, \dots, \frac{d^{m-1}u}{dz^{m-1}}, b_1, \dots, b_k\right) = \tilde f(t, x^1, \dots, x^{m+1}, b_1, \dots, b_k)\Big|_{t=z, x^1=u, x^2=\frac{du}{dz}, \dots, x^m=\frac{d^{m-1}u}{dz^{m-1}}}. \tag{29}$$

Thus, we have obtained a transformation relating the ODEs (24) and (28), and this transformation preserves the integrability properties.

In view of the particular interest in the study of ODEs let us restate Theorem 1 for this special case directly in terms of ODEs. To this end we first recall the relevant definitions following [14].

A generalized vector field $Y = h(z, u, du/dz, \dots, d^{m-1}u/dz^{m-1})\partial/\partial u$ is a *(generalized) symmetry* for (24) if we have

$$D^m(h) - \sum_{j=0}^{m-1} \frac{\partial F}{\partial u_j} D^j(h) = 0.$$

Here $u_0 \equiv u, u_j \equiv d^j u/dz^j$, and we have introduced the so-called operator of the total z-derivative

$$D = \frac{\partial}{\partial z} + F\frac{\partial}{\partial u_{m-1}} + \sum_{j=0}^{m-2} u_{j+1}\frac{\partial}{\partial u_j}$$

(here we treat z and u_j as formally independent entities, see e.g. [14] for details).

Also, a function $I = I(z, u, du/dz, \dots, d^{m-1}u/dz^{m-1})$ is a *(first) integral* for (24) if $D(f) = 0$.

It is easily seen that upon passing from (24) from (26) an integral of motion I and the *prolongation* (see e.g. [14]) of a symmetry Y

$$\mathrm{pr}\, Y = \sum_{j=0}^{m-1} D^j(h)\frac{\partial}{\partial u_j}$$

become respectively an integral and a symmetry for (24) in the sense of the definitions from Section 2.

If $Y_i = h_i(z, u, du/dz, \dots, d^{m-1}u/dz^{m-1})\partial/\partial u$, $i = 1, 2$, are two symmetries for (24) in the sense of the above definition, their commutator is [14] given by the formula

$$[Y_1, Y_2] = (\operatorname{pr} Y_1(h_2) - \operatorname{pr} Y_2(h_1))\, \partial/\partial u, . \tag{30}$$

and of course it is again a symmetry for (24).

With all this in mind we are ready to state the ODE version of Theorem 1.

Corollary 3 *Under the above assumptions, let (24) be an ODE whose right-hand side depends on k parameters $a_1, \dots, a_k$, and let (24) have k functionally independent integrals $I_1, \dots, I_k$ such that (5) holds.*

Then the following claims hold:

i) the functions $\tilde{I}_i$, $i = 1, \dots, k$, are functionally independent integrals for the transformed ODE (28), and we have

$$\det\left(\|\partial \tilde{I}_i/\partial b_j\|_{i,j=1,\dots,k}\right) \neq 0;$$

ii) if $J_1, \dots, J_m$ is another set of integrals for (24) such that all integrals $I_1, \dots, I_k$, $J_1, \dots, J_m$ are functionally independent, then $\tilde{I}_1, \dots, \tilde{I}_k, \tilde{J}_1, \dots, \tilde{J}_m$ are functionally independent integrals for (28);

iii) if $Y_1, \dots, Y_r$ are linearly independent generalized symmetries for (24) such that

$$\operatorname{pr} Y_p(I_j) = 0 \quad \textit{for all} \quad p = 1, \dots, r \quad \textit{and} \quad j = 1, \dots, k,$$

then $\tilde{Y}_1, \dots, \tilde{Y}_r$ are linearly independent generalized symmetries for (28), and

$$\operatorname{pr} \tilde{Y}_p(\tilde{I}_j) = 0 \quad \textit{for all} \quad p = 1, \dots, r \quad \textit{and} \quad j = 1, \dots, k;$$

iv) if under the assumptions of (ii) and (iii) the symmetries $Y_1, \dots, Y_s$, where $s \leq r$, span an involutive distribution, i.e., $[Y_p, Y_q] = \sum_{g=1}^{s} c_{pq}^g(a_1, \dots, a_k, I_1, \dots, I_k, J_1, \dots, J_m) Y_g$ for all $p, q = 1, \dots, s$ (the commutator is now given by (30)!), then the symmetries $\tilde{Y}_1, \dots, \tilde{Y}_s$ also span an involutive distribution, $[\tilde{Y}_p, \tilde{Y}_q] = \sum_{g=1}^{s} \tilde{c}_{pq}^g \tilde{Y}_g$, for all $p, q = 1, \dots, s$, where $\tilde{c}_{pq}^g = c_{pq}^g(\tilde{I}_1, \dots, \tilde{I}_k, b_1, \dots, b_k, \tilde{J}_1, \dots, \tilde{J}_m)$. If $c_{pq}^g \in \mathbb{K}$ are constants (in particular, they do not depend on a_j, $j = 1, \dots, k$), and thus Y_g, $g = 1, \dots, s$, form a Lie algebra, then $\tilde{Y}_g$, $g = 1, \dots, s$, form an isomorphic Lie algebra.

Example 4. Consider equation 6.45 from [9]

$$\frac{d^2u}{dz^2} = c\left(\frac{du}{dz}\right)^2 + a$$

which admits an integral of the form

$$I = ((du/dz)^2 + (a(1 + 2cu))/(2c^2)) \exp(-2cu).$$

Let $a_1 \equiv a$ and $b_1 \equiv b$. Then we have

$$\tilde{I} = 2c^2(b \exp(-2cu) - (du/dz)^2)/(1 + 2cu),$$

which is an integral for the transformed equation,

$$d^2u/dz^2 = c(du/dz)^2(1 - 2c^2/(1 + 2cu)) + 2bc^2 \exp(-2cu)/(1 + 2cu).$$

Just as for dynamical systems (1), for ODEs we also often can add parameters by hand through changes of variables, and apply the generalized Stäckel transform to the resulting equations.

Example 5. Consider equation 7.7 from [9],

$$\frac{d^3u}{dz^3} = \frac{1}{u}\frac{d^2u}{dz^2}\frac{du}{dz} - u^2\frac{du}{dz},$$

which admits an integral of the form

$$\frac{1}{u}\frac{d^2u}{dz^2}+\frac{u^2}{2},$$

and rescale $u\to au$. This yields the equation

$$\frac{d^3u}{dz^3}=\frac{1}{u}\frac{d^2u}{dz^2}\frac{du}{dz}-au^2\frac{du}{dz}$$

with an integral

$$I=\frac{1}{u}\frac{d^2u}{dz^2}+\frac{a^2u^2}{2}.$$

Again let $a_1\equiv a$ and $b_1\equiv b$. Then we have

$$\tilde{I}=\left(2\left(b-\frac{1}{u}\frac{d^2u}{dz^2}\right)\right)^{1/2}\frac{1}{u},$$

which is an integral for the transformed equation,

$$\frac{d^3u}{dz^3}=\left(\frac{1}{u}\frac{d^2u}{dz^2}-\left(2\left(b-\frac{1}{u}\frac{d^2u}{dz^2}\right)\right)^{1/2}u\right)\frac{du}{dz}.$$

7 Conclusions and discussion

In this paper we extend the multiparameter generalized Stäckel transform, or the coupling constant metamorphosis, to general dynamical systems (1) and ODEs and studied the properties of this extension. In particular, we present sufficient conditions under which the transformed system inherits the integrability properties of the original one: the existence of Lax representation, (sufficiently many) integrals of motion and symmetries, etc. In contrast with the Hamiltonian case [17], for general dynamical systems (1) we can avoid introducing the reciprocal transformation for (the solutions of) the equations of motion. As a result, the relationship among the solutions of the original system (1) and the transformed system (8) is much simpler than in the Hamiltonian case. Note that the same approach was successfully applied to discrete dynamical systems, see [16] for details.

Our results naturally lead to a number of open problems related to the generalized Stäckel transform, of which we list below just a few.

First of all, it would be very interesting to find out (both in the Hamiltonian and the non-Hamiltonian case) when the transformed dynamical system is *algebraically* [19, 20] completely integrable provided so is the original system. On a related note, the study of relationship among the differential Galois groups of the variational equations (see e.g. [18, 2, 12] and references therein for the relevant definitions) for original and transformed systems would be of interest too.

Second, we have just barely scratched the surface by noticing in Proposition 1 that the transformed system inherits the existence of a Lax representation from the original system, and e.g. understanding what is the precise relationship among the Darboux [13] and Bäcklund [10, 11] transformations for the original and transformed system would certainly be worth the while.

Third, it is highly desirable to study more systematically the issue of when inserting the parameters ‘by hand’ (cf. the above Examples 2 and 5) and subsequent transforming of the resulting systems leads to interesting new examples. We expect this technique to yield a significant extension of the pool of exactly solvable dynamical systems and ODEs. A good starting point here could be e.g. to find the transformed counterparts for the ODEs that linearize on differentiation [6].

Finally, the simplest but perhaps also the most important (cf. e.g. [17] for the Hamiltonian systems) special case when integrals of motion are *linear* in the parameters undoubtedly deserves to be explored in far more details.

We hope that the present paper will stimulate further research in these and related areas.

Acknowledgments

This research was supported in part by the Ministry of Education, Youth and Sports of the Czech Republic (MŠMT ČR) under grant MSM 4781305904 and by the Czech Grant Agency (GA ČR) under grant P201/11/0356.

It is my great pleasure to thank Prof. M. Błaszak, Prof. A. Maciejewski, and Prof. V.B. Matveev for stimulating discussions, and Dr. R.O. Popovych for reading the manuscript of the present paper and making many helpful comments. I also thank the referees for useful suggestions.

References

[1] M. Ablowitz, H. Segur, Solitons and the inverse scattering transform, SIAM, Philadelphia, PA, 1981.

[2] M. Ayoul and N.T. Zung, Galoisian obstructions to non-Hamiltonian integrability, Comptes Rendus Math. **348** (2010), no.2324, 1323–1326, arXiv:0901.4586

[3] M. Błaszak and A. Sergyeyev, Generalized Stäckel systems, Phys. Lett. A **375** (2011), no.27, 2617–2623.

[4] O.I. Bogoyavlenskij, Extended integrability and bi-Hamiltonian systems, Comm. Math. Phys. **196** (1998) 19–51.

[5] C.P. Boyer, E.G. Kalnins, and W. Miller, Jr., Stäckel-equivalent integrable Hamiltonian systems, SIAM J. Math. Anal. **17** (1986) 778–797.

[6] E.V. Ferapontov and S.R. Svirshchevskii, Ordinary differential equations which linearize on differentiation, J. Phys. A: Math. Theor. **40** (2007) 2037–2043, arXiv:nlin/0608062

[7] A. Goriely, Integrability and nonintegrability of dynamical systems, World Scientific, River Edge, NJ, 2001.

[8] J. Hietarinta, B. Grammaticos, B. Dorizzi, and A. Ramani, Coupling-constant metamorphosis and duality between integrable Hamiltonian systems, Phys. Rev. Lett. **53**, 1707–1710 (1984).

[9] E. Kamke, Differentialgleichungen: Lösungmethoden und Lösungen, Band I: Gewöhnliche Differentialgleichungen, 6 Aufl., Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1959.

[10] V.B. Kuznetsov and E.K. Sklyanin, On Bäcklund transformations for many-body systems. J. Phys. A: Math. Gen. **31** (1998) 2241–2251, arXiv:solv-int/9711010

[11] V. Kuznetsov and P. Vanhaecke, Bäcklund transformations for finite-dimensional integrable systems: a geometric approach, J. Geom. Phys. **44** (2002) 1–40, arXiv:nlin/0004003

[12] A.J. Maciejewski and M. Przybylska, Differential Galois theory and integrability, Int. J. Geom. Methods in Mod. Phys. **6** (2009), no. 8, 1357–1390, arXiv:0912.1046

[13] V. B. Matveev and M.A. Salle, Darboux transformations and solitons, Springer, Berlin, 1991.

[14] P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed., Springer, N.Y., 2000.

[15] A.D. Polyanin and V.F. Zaitsev, Handbook of exact solutions for ordinary differential equations, 2nd ed., Chapman & Hall/CRC, Boca Raton, FL (2003).

[16] J.A.G. Roberts, A. Iatrou, and G. R. W. Quispel, Interchanging parameters and integrals in dynamical systems: the mapping case, J. Phys. A: Math. Gen. **35** (2002) 2309–2325.

[17] A. Sergyeyev and M. Błaszak, Generalized Stäckel transform and reciprocal transformations for finite-dimensional integrable systems, J. Phys. A: Math. Theor. **41** (2008) 105205, arXiv:0706.1473

[18] M. van der Put and M.F. Singer, Galois theory of linear differential equations, Springer, Berlin, 2003.

[19] P. Vanhaecke, Algebraic integrability: a survey, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **366** (2008), no. 1867, 1203–1224.

[20] P. Vanhaecke, Intégrabilité algébrique: une introduction, Afr. Diaspora J. Math. (N.S.) **9** (2010) 1–16.